

Swarms, Phase Transitions, and Collective Intelligence; and a Nonequilibrium Statistical Field Theory of Swarms and Other Spatially Extended Complex Systems

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Swarms, Phase Transitions, and Collective Intelligence

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Abstract

A spacially extended model of the collective behavior of a large number of locally acting organisms is proposed in which organisms move probabilistically between local cells in space, but with weights dependent on local morphogenetic substances, or morphogens. The morphogens are in turn affected by the passage of an organism. The evolution of the morphogens, and the corresponding flow of the organisms constitutes the collective behavior of the group. Such models have various types of phase transitions and self-organizing properties controlled both by the level of the noise, and other parameters.

The model is then applied to the specific case of ants moving on a lattice. The local behavior of the ants is inspired by the actual behavior observed in the laboratory, and analytic results for the collective behavior are compared to the corresponding laboratory results.

It is hoped that the present model might serve as a paradigmatic example of a complex cooperative system in nature. In particular swarm models can be used to explore the relation of nonequilibrium phase transitions to at least three important issues encountered in artificial life. Firstly, that of emergence as complex adaptive behavior. Secondly, as an exploration of continuous phase transitions in biological systems. Lastly, to derive behavioral criteria for the evolution of collective behavior in social organisms.

1 Introduction

1.1 The appeal of swarms

The swarming behavior of social insects provides fertile ground for the exploration of many of the most important issues encountered in artificial life.

Not only do swarms provide the inspiration for many recent studies of the evolution of cooperative behavior[6, 19, 20], but the action of the swarm on a scale of days, hours, or even minutes manifests a near constant flow of emergent phenomena of many different types[1, 2, 8, 11, 14, 17, 26, 27]. Models of such complex behavior range from abstract cellular automata of models[21] to more physically realistic computational simulations.[7, 12] The notion that complex biological behavior, from the molecular to the ecological, can be the result of parallel local interactions of many much simpler elements is one of the fundamental themes of artificial life.[21] The swarm, which is a collection of simple locally interacting organisms with global adaptive behavior, is a quite appealing subject for the investigation of this theme.

When one includes an evolutionary dimension the appeal becomes even more robust, since we have, in many ways, a much better notion of the ultimate purpose or utility of insect behavior than we have of many other types of emergent phenomena in nature. The notion of utility provides a link between the emergent *behavior* of swarms, and the *evolution* of cooperative social behavior.

An additional impetus to this type of study which is lacking in many areas of artificial life research is the contact with experiment so necessary for the healthy growth of science. By this I mean not just computer simulations, but actual work with real organisms.[1, 8, 26] In many ways the physical motivations behind the types of models discussed here are inspired by such experiments. The knowledge that many kinds of controlled investigations can be performed on systems which closely resemble the models described here not only informs the interpretation of the results, but suggests new types of laboratory studies.

In the end perhaps the most pervasive appeal of swarms centers on a kind of emotional attractiveness of the subject. Undoubtedly all of the above considerations play a role in this, but probably the main reason is hidden within in the human psyche. More than a paradigm, swarms are almost, at times, an archetype.

1.2 Basic principles of swarm intelligence

It is useful to list some broad behavioral categories which might be classified as collective intelligence, or swarm intelligence. These may be thought of as evolutionary principles of selection, and are not intended to be definitive.

The first is the **proximity principle**. The group should be able to do elementary space and time computations. Since space and time translate

into energy expenditure,[29] the group should have some ability to compute the utility of a given response to the environment in these terms. Computation is understood as *a direct behavioral response to environmental stimuli which in some sense maximizes the utility to the group as a whole of some type of activity*. Although the kinds of activity may vary greatly, depending on both the type and complexity of the organisms, some typical activities include the search and retrieval of food, the building of nests, defense of the group, collective movement, and in the case of higher organisms, the interaction necessary for many social functions.[17]

Second is the **quality principle**. The group should be able to respond not only to time and space considerations, but to quality factors, for instance, to the quality of foodstuffs or safety of location.

Third is the **principle of diverse response**. The group should not allocate all of its resource along excessively narrow lines. It should seek to distribute its resources along many modes as insurance against the sudden change in any one of them due to environmental fluctuations. It is clear that a completely ordered response to the environment, even if possible, may not even be desirable.

Fourth is the **principle of stability**. The group should not shift its behavior from one mode to another upon every fluctuation of the environment, since such changes take energy, and may not produce a worthwhile return for the investment. The other side of the coin is the **principle of adaptability**. When the rewards for changing a behavioral mode are likely to be worth the investment in energy, the group should be able to switch. The best response is likely to be a balance between complete order, and total chaos, and therefore, the level of randomness in the group is an important factor. Enough noise will allow a diverse response, while too much will destroy any cooperative behavior.

The behavior of many complex adaptive systems would probably fall into some versions of these principles. It is amusing to note the resemblance of these rules to many good economic decision making principles, or folk maxims like *time is money, only buy the best, don't put all your eggs in one basket, better safe than sorry, a bird in the hand is worth two in the bush, invest for the future*, etc.[13]

1.3 Inspiration: the behavior of real ants

The techniques used in some recent experiments with ants[7, 8, 14, 26] in many ways inspired the approach used here. Typically, ants in the labora-

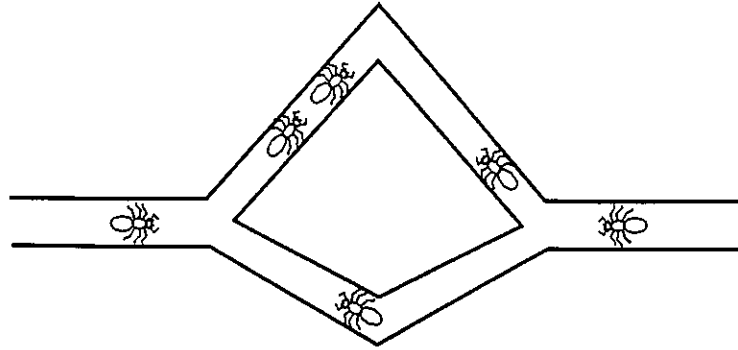


Figure 1: The binary bridge experiment.

tory are exposed to a set of bridges connecting two or more areas where the ants explore, feed, and go about their business. As the ants wander they discover and cross the bridges. As they move on the effectively one-dimensional paths they come to junctions where they choose a new branch, and continue on their way. Figure 1 shows a picture of what such a bridge might look like. Since the ants both lay and follow scent as they walk, the flow of ants on the bridges changes as time passes. For instance, in the case shown above, most of the flow will eventually concentrate on the one of the branches. In this case the swarm is said to have chosen a branch of the bridge. The types of emergent behaviors of the ants can then be studied in a controlled manner by observing their response to various situations. In addition, many interesting mathematical models and computer simulations have been studied which capture some of the behavior observed in the laboratory[1, 7, 27].

The experimental results indicate the type of environmental, or **emergent computation** real ants perform. The term environmental computation here again refers to the fact that the ants collectively perform information gathering and processing on the local environment. Both the information gathering and processing happen simultaneously, and without centralized controls. For example, both the location of a food source and its utilization are *computed* by the self-organization of a column of ants between the nest and the food source. These experimental observations form a basic set of tasks any model needs to explore. The following results were obtained using a few different species of ants[8, 14, 26].

1. When ants are exposed to two paths of unequal length the ants will choose the shortest path.
2. If a shorter path is offered after the ants have chosen, they are unable to

switch to the new path.

3. The ants will break symmetry and chose one path, even when both paths are equal.
4. If ants are offered two unequal food sources they will usually choose the richest source.
5. If a richer food source is offered after the ants have chosen, some species can switch to this new source, and others are unable to.
6. If two equal food sources are offered the ants exploit the source unequally, breaking symmetry.

In addition, the rules for the effect of the pheromone density on the motion of the ants can be determined experementally from a setup such as the one pictured in Figure 1. It has been found from such experiments that ants choose a branch of the bridge in proportion to the function $(\alpha + \sigma)^\beta$, where σ is the density of pheromone on the branch measured in unit of the average density layed down on passage of a single ant, and α is a fixed parameter with the same units. β is some dimensionless fixed parameter. For one type of ant, the argentine ant, values of $\alpha = 20$ and $\beta = 2$ were obtained.

1.4 Goals

The collective behavioral characteristics of a group of organisms must, of course, be encoded in the behavior of the individual organisms. We will be interested in *how* the collective behavior is encoded by the individual behavior. In particular, we explore the idea that complex adaptive behavior is the result of the *interactions between* organisms as distinct from behavior which is a direct result of the actions of *individual* organisms.

In line with the ideas proposed above, the most important modeling constraint in what follows is the **principle of locality**. In the models we will study the behavior of the individual organisms will be determined solely by local influences. This means that the individual organisms will not have any memory, non-local navigational skills, or any type of behavior that involves storage of internal information. Any information flow must then be a product of the collective behavior. Wilson[32] introduced the concept of mass communication to designate information transmission which cannot be communicated by the individual. Here we extend the designation to include

transmission of information the individuals can not even possess. Of course the actual situation in nature will almost always be more complicated than this, but we may hope to achieve some level of understanding by studying such restricted models, and comparing them to real situations. In this way it may be possible to isolate many of the collective emergent processes which are taking place.

1.5 Preview

In section 2 the swarm network model is presented, and its analogy with other connectionist models is discussed. We show that the distribution of organisms on the network always relaxes, on the short time scale, to a unique stationary distribution which is independent of the initial configuration. This allows us to write down the general deterministic learning rules for the network. In section 3 we introduce a type of dynamics, inspired by experiment, which allows us to make an analogy with a thermodynamic system. The quite general conditions for self-organization (symmetry breaking) are derived, and the various attractors and phase transitions of the swarm network system are explored. In section 4 a specific model of ant swarms is presented. The behavioral function of the ants is taken from the experimental results, and a theoretical model. I will particularly focus on the role of phase transitions, which are of relevance to the study of artificial life in at least three important ways. The first and most obvious role is that phase transitions control the global behavior of the swarm. In section 5 a few simple cases are compared to the observed behavior of ants in the laboratory. Secondly, since it has recently been suggested[22] that second order phase transition might have an important influence on the complex adaptive, and computational properties of biological systems, these transitions are worthy of study in their own right. Here we provide an example inspired by real-life biology which can be studied. Finally, not all possible phase transitions have significance on a behavioral level. Phase transitions on variation of the behavioral parameters of the organisms, which are fixed on the behavioral time scale, can provide *evolutionary criteria* for the development of cooperative behavior in social organisms. We thus hope to illuminate this issue as well.

2 Swarm networks

In this section we present the basic structure and dynamics of swarm networks. The dynamics of the organisms is discussed, and the deterministic learning rule for a swarm is derived.

2.1 Connectionist models

Connectionist models[9] have three basic earmarks. Firstly, their **structure** consists of a discrete set of *nodes*, and a specified set of *connections* between the nodes. For example, neural networks, the archetypal connectionist systems, are composed of neurons (nodes), and the neurons are usually linked by synapses (connections). Secondly, there are the **dynamics** of the *nodal variables*. In the case of neural networks the nodal variables are the firing rates of each neuron. The dynamics are controlled by the connection strengths, and the input-output rule of the individual neurons. The dynamics of the whole system is the result of the interaction of all the neurons. Lastly there is **learning**. In its most general sense learning describes how the connection strengths, and hence the dynamics, evolves. In general there is a separation of time scales between dynamics and learning, where the dynamical processes are much faster than the learning processes.

In addition to neural networks there are many other type of connectionist type models, such as Autocatalytic chemical reactions, Classifier systems, and Immune networks[10, 28], to mention just a few. Swarm networks are just another such example.[24]

2.2 Network structure

The structure of the network is a representation of the physical space on which the organisms move, and possibly of certain environmental factors and constraints. In this paper we will not seek detail on arbitrarily small scale, but will divide the physical space into discrete elements, or **nodes**, which will be labeled by an index $i = 1, \dots, m$. In certain cases this discretization may be a natural reflection of the actual physical setup of a laboratory experiments such as those discussed above, or another naturally occurring discrete structure such as an existing network of path segments. In other cases the discretization may represent a more abstract division of the physical space. For instance, positions on a plane may be divided up into a chess board of discrete cells. Additionally, each discrete position in space

might have additional discrete divisions representing the possible orientations of an organism at that point. Each discrete position and orientation will be specified by a single node. A network is then a kind of discretized phase space on which the organisms move.

In addition to nodes, we must specify **connections**. Connections express the possibility of an organism moving from one node to another, and are specified by an ordered pair of nodes, one of which is the initial node, and one of which is the final node. Obviously, most transitions will not be possible. The set of possible final nodes for an initial node are said to be the nodes *local* to the initial node.

Lastly we must specify boundary conditions on the network. These will take the form of **adsorbing sites**, and **input sites**. Adsorbing sites are nodes from which an organism may leave the network entirely. Input sites are nodes which receive an input of organism from the world outside the network. For the purposes of this paper we will consider only closed networks.

2.3 Dynamic and parametric network variables

A number of organisms will be allowed to move on the network. Most of what follows can easily be extended to the case where there are a number of different *types* of organisms, but for simplicity we will only deal with the case where there is one type present. Each node has a given volume or measure, μ_i , with units depending on the dimension of the nodes. We will denote the number of organisms at node i at time t by $n^i(t)$, so the density of organism at i is $\rho^i(t) = n^i(t)/\mu_i$.

In addition there is a quantity of a morphogenetic substance, or morphogen, at each node. I have adopted this terminology from Turing's famous paper, *The Chemical Basis of Morphogenesis*[30], where the term is taken to indicate form producer. While in the present case there is not a one-to-one relationship with Turing's morphogen concept, the basic idea is the same. The term is adopted mainly to avoid the misunderstanding that the processes modeled here depend on any specific realizations or mechanisms. Much of what follows has quite wide applicability.

The density of morphogen on node i will be denoted σ^i . We will for simplicity consider the case where there is only one *type* of morphogen present. This may be thought of as chemical substance which the organisms both emit, and respond to, but it is possible in certain circumstances it might have another physical meaning. For instance, such networks could model the formations of animal trails, or routes in the undergrowth. The density

of plant growth at a given point would be the analogous to a morphogenetic substance. The route that animals take while moving from place to place is influenced by the density of undergrowth. In addition, the undergrowth is affected by the passage of animals, which tend to crush young plants, and otherwise impede the growth in that area. One might expect the result, over time, to be the formation of trails. This type of consideration is also known to play a role in the formation of more or less permanent network of *physical* trails by insect colonies.[17]

ρ^i and σ^i are essentially the dynamical variables of the model. However, we can, if we wish, make a distinction between **dynamical** variables, and **parametric** variables under certain conditions. For practical purposes we will assume that the σ^i are more or less constant for time scales typical of the variables ρ^i . We will then treat the σ^i as parameters which determine the dynamics of the ρ^i . The ρ^i are said to be *slaved* to the σ^i . They are parametric *variables* because they are endowed with a dynamics of their own. This dynamics is usually known in connectionist terminology as *learning*. This separation of time scales, and the resulting distinction of dynamic versus parametric variables is a common theme running through connectionist theories. For some types of behavior we may not be able to strictly separate time scales, but we will view these situations as complications on the base of a model where the time scales are well separated. In fact, it is the passage of the swarm through a supercritical point of its global dynamics which makes possible the amplification of small nonequilibrium signals, such as time delay feedback.[8] A good example of this will be our analysis of the choice of a closer food source in section 5.

In addition to dynamic and parametric variables, there will also be **fixed parameters** which can be considered constant on both dynamical and learning time scales. The network will possess certain parameters such as the measure of each node, μ_i , or the orientation of each node at the junction if we are modeling a network of paths. In addition the organism may have internal parameters which determine their response to, and emission of, the morphogenic substances. The choice of these parameters by natural selection may be considered an additional type of dynamics of the system. Lastly, there may be one or more parameters which describe the evolution of the parametric variables, such as the decay rate of the morphogen.

2.4 Dynamics: the flow of organisms

In addition to reacting to the morphogens, the organisms may in general interact with each other. For the purpose of this paper we will consider situations where the organism do not interact directly, but only through the medium of the morphogenetic substance. In this case the the flow on a network is described by

$$\mu_i \frac{d\rho^i}{dt} = \sum_j (v_0 \omega_{ij} \rho^j - v_0 \omega_{ji} \rho^i) - v_0 a^i \rho^i + f^i, \quad (1)$$

where ω_{ij} is the probability that organisms leaving node j will choose node i , and a^i is an adsorption probability, that is, the probability that organisms leaving i will leave the network entirely. v_0 is a rate constant which can be thought of as the speed of the organisms. The left hand side of the equation is just the rate of change of the number of organisms on node i . The first term on the right hand side represents the flow of organisms into the node from all of the connecting nodes. The second term is the flow out of the node to all the local nodes. The third term represents the loss of organisms to the outside world at an adsorbing site, and the last term is the number of organisms entering node i per unit time from some outside source. Clearly $\omega_{ij}, a^i, f^i(t) \geq 0$. Additionally, probability considerations lead to the following restriction,

$$a^i + \sum_j \omega_{ji} = 1, \quad (2)$$

which can be read as, the probability that an organism on node i goes somewhere is one. The ω_{ji} will generally depend on the parametric variables, and also on the fixed parameters of the network, and a^i and f^i may be taken to be fixed, or given some special dependence on the parametric variables depending on the situation we are trying to model.

The distribution of organisms on the network described above in nearly all physically reasonable situations relaxes to a stationary state which is independent of the initial distribution. The state is known as a stationary state because the distribution is constant in time. Since in general the flow of organisms at any given point on the network does not vanish, the state is not an equilibrium state. The more specific case of equilibrium states, and the conditions necessary to produce them will be discussed in detail in the next section.

2.5 Learning: the parametric dynamics

The learning rule is a prescription for the way the morphogen distribution changes with time. It is not arbitrary, but depends on the details of the mechanism by which the morphogens evolve. For our purpose we will make use of the following simple, and sensible rule. As time passes, the morphogen density decays in proportion to the amount present with decay rate κ . In addition, each organism at i lays down a density η_i of the morphogen as it passes a point. The evolution of the morphogen distribution on the network is then described by the equation,

$$\frac{d\sigma^i}{dt} = -\kappa \sigma^i + \eta_i \rho^i, \quad (3)$$

where σ^i is the density of the morphogen on node i , and ρ^i is the density of organism on node i . The Eqs. (1) and (3) then completely describe the dynamics once the dependence of the ω_{ij} on the σ^i is specified.

As discussed before, we will make the assumption of separation of time scales. In this case the flow of organisms relaxes to a stationary state on time scales short compared to the typical times scales of Eq. (1). The **deterministic approximation** then consists in replacing the ρ^i by their stationary values $\rho_s^i(\sigma)$ and in *ignoring the fluctuations* about these values. This leads us to the generalized learning rule for a swarm network

$$\frac{d\sigma^i}{dt} = -\kappa \sigma^i + \eta_i \rho_s^i(\sigma). \quad (4)$$

It is of interest to note that even though we have strictly local couplings between nodes, the learning rule is a globally coupled system of equations, since $\rho_s^i(\sigma)$ depends in a complicated way on the parametric variables σ^i . This is made possible by the separation of time scales. On the dynamical time scale the system *feels out* its entire space of configurations, and relaxes to a state which is determined by global considerations, even though the dynamics is strictly local. This globally determined state then determines the dependence of some of the variables in the local learning rule. *Globally coupled parametric dynamics in strictly local models via separation of time scales* is an important aspect of connectionist models, and is one of the central reasons why connectionism is such an attractive model of adaptive complex systems.

3 The thermodynamic analogy

3.1 Detailed balance and the thermodynamic analogy

For the rest of this paper we will consider the case where the transition matrix takes the form $\omega_{ij} \propto f(\sigma^i) \Omega_{ij}$, where f is some weighting function describing the effect of the σ^i on the motion of the organisms, and $\Omega_{ij} = 1$ if i connects to j , and zero otherwise. When properly normalized the transition matrix is given by

$$\omega_{ij} = \frac{f(\sigma^i) \Omega_{ij}}{\sum_k f(\sigma^k) \Omega_{kj}}. \quad (5)$$

Transition matrices of this type obey the detailed balance relations $\omega_{ij} f(\sigma^j) = \omega_{ji} f(\sigma^i)$. The property of detailed balance allows us to determine all the statistical properties of the quasi-stationary particle field. In addition a one-to-one analogy with a thermodynamic system with energy $U(\sigma)$ and temperature $T = \beta^{-1}$ can be made if we set $f(\sigma) = \exp(-\beta U(\sigma))$, where any parameter T which affects f can be regarded as a temperature parameter if $f(\sigma; T)$ scales like $f(\sigma; \alpha T) = f^{-\alpha}(\sigma; T)$. Statistical quantities of interest can then be calculated from the one-particle partition function

$$Z = \frac{1}{V} \sum_i \mu_i \exp(-\beta U(\sigma^i)) \quad (6)$$

according to the usual prescriptions,[18] where V is the total volume of the system. The N -particle partition function is $Z_N = Z^N$.

Let us partially evaluate the partition function over the volume μ_ϵ of the network with a given energy ϵ ,

$$Z = \frac{1}{V} \sum_{U \neq \epsilon} \mu_i \exp(-\beta U(\sigma^i)) + \frac{\mu_\epsilon}{V} \exp(-\beta \epsilon). \quad (7)$$

The mean equilibrium particle density in the energy state ϵ is then given by[18]

$$\rho_\epsilon^e = \frac{N}{VZ} \exp(-\beta \epsilon). \quad (8)$$

3.2 The order parameter equation

The equations for the pheromonal field densities can be written

$$\frac{ds^i}{d\tau} = -s^i - \rho_e^i(s), \quad (9)$$

where we have introduced the new variables $\tau = \kappa t$, and $s^i = \kappa \sigma^i / \eta$. We have adiabatically eliminated the organisms from the picture by replacing ρ^i with ρ_e^i . We introduce the bimodal variables

$$s^+ = \frac{1}{\mu^+} \sum_{i; s^i > \gamma} s^i, \quad s^- = \frac{1}{\mu^-} \sum_{i; s^i < \gamma} s^i \quad (10)$$

where $\gamma = N/V$ is the mean density of organism on the network, and

$$\mu^+ = \sum_{i; s^i > \gamma} \mu^i, \quad \mu^- = \sum_{i; s^i < \gamma} \mu^i. \quad (11)$$

We will make use of the mean-field approximation $s^i = s^\pm$, which leads to the mean-field bimodal equations

$$\frac{ds^+}{d\tau} = -s^\pm + \rho_e^\pm(s^+, s^-). \quad (12)$$

The quasi-stationary distribution $\rho_e^\pm(s^+, s^-)$ is calculated from the master equation for the particle density. In the case of a closed systems with detailed balance

$$\rho_e^\pm(s^+, s^-) = -\frac{N}{\beta \mu^\pm} \frac{\partial \ln Z}{\partial \epsilon^\pm}. \quad (13)$$

It will be helpful to introduce the dimensionless parameter $v = \mu^- / \mu^+$, the ration of the volume in the - state to the volume in the + state. For compactness we also define the function

$$R(s^+, s^-) = f(s^+) / f(s^-). \quad (14)$$

The equations can then be put in the form

$$\frac{ds^+}{d\tau} = -s^+ + \frac{\gamma(1+v)R^\beta}{v + R^\beta}, \quad (15)$$

$$\frac{ds^-}{d\tau} = -s^- + \frac{\gamma(1+v)}{v + R^\beta}. \quad (16)$$

Instead of working in the variables s^\pm we will chose the new order parameter variables

$$m = s^+ - s^-, \quad s = \frac{s^+ + v s^-}{1+v} - \gamma. \quad (17)$$

The second of these is proportional the difference between the total density of pheromone present and the value at which the total density of pheromone equilibrates. The equation for the evolution of s is given by $ds/d\tau = -s$, which has solutions $s \propto e^{-\tau}$. No self organization will occur until the pheromone density has built up sufficiently. Thus the pheromone density will initially evolve uniformly over the configuration space, there will be little feedback, and m will be small on time scales where $s \rightarrow 0$. We can thus make the adiabatic approximation $s = 0$. The deterministic equation for the order parameter m is then given by

$$\frac{dm}{d\tau} = -m + F(m), \quad (18)$$

where

$$F = \gamma(1 + \nu) \frac{R^\beta - 1}{R^\beta + \nu}, \quad (19)$$

and where the F is determined as a function of m by

$$R(m) = R\left(\gamma + \frac{\nu m}{1 + \nu}, \gamma - \frac{m}{1 + \nu}\right). \quad (20)$$

In addition the longtime evolution of the order parameter will only depend on the longtime limit $s \rightarrow 0$ and is not affected by the validity of the adiabatic approximation.

The deterministic dynamics is then described by the equation

$$\frac{dm}{d\tau} = -\Psi'(m), \quad (21)$$

where the deterministic potential $\Psi(m)$ is given by

$$\Psi(m) = \frac{1}{2}m^2 - \int^m F(x) dx. \quad (22)$$

3.3 Phase transitions

We will be interested in the various phase transitions in the collective behavior of the swarm. These are the points where the behavior changes abruptly upon variation of some parameter. As mentioned earlier, these points determine the behavior of the swarm when the parameter that is varied is an external one. In this case the basins of attraction of Ψ will determine the

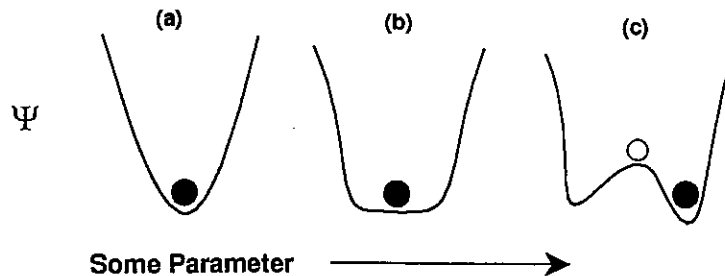


Figure 2: A phase transition.

behavior of the swarm, so a phase transition represents a behavioral transition. Phase transitions are also significant in establishing evolutionary criteria when the parameter which is varied is a *fixed* behavioral parameter.

The values of the order parameter m_k are determined from the condition

$$\Psi'(m_k) = 0, \quad (23)$$

and their stability by

$$\Psi''(m_k) > 0. \quad (24)$$

That is, by the the minima of Ψ . When a parameter is varied, the shape of Ψ varies. A phase transition occurs when there is a transition, such as the one shown in Figure 2 from one type of shape to another. In this case the various macroscopic states of the swarm may change in an abrupt way. The point of phase transition is known as a critical point, illustrated by the shape in Figure 2(b).

To include some basic terminology, we usually distinguish two types of phase transitions, **first order** and **second order**. First order designates phase transition where the macroscopic states change in a discontinuous way upon passage through the critical point, and second order designates phase transitions where the states change in a continuous way. Second order phase transitions usually have some kind of symmetry which is broken when passing through the critical point. This symmetry breaking is caused by the natural fluctuations in the system which we neglected in the deterministic approach. It is at the critical point where we would expect the fluctuations to become the most pronounced, and our deterministic analysis to have the greatest problems.

When a system passes through a second order transition, it may be left sitting at the top of the hill in the center of Figure 2(c). At this point the system is like a pencil balanced on its end. We cannot tell which direction it

will fall, but a small perturbation can send it falling in a certain direction. It is the sensitivity of the system at this point to external influences which makes them of interest as a kind of *information amplifier* in biological systems. Typically, susceptibilities, the quantities which measure the response of a system to external influence, become infinite at the critical point.

In addition, many macroscopic quantities scale in a characteristic way near a critical point, according to **critical exponents**. These exponents have, in many cases, universal values which are independent of the details of the given universality class of models being studied.

From conditions 23 and 24 we can determine the stability condition for the homogeneous state, $s^i = \gamma$,

$$\gamma \chi(\gamma) < T, \quad (25)$$

where $T = 1/\beta$ is the temperature parameter, and where χ is known as the chemotatic factor, or force $\chi(x) = -U'(x)$.

4 Ant swarms

4.1 Microscopic ant behavior

The microscopic dynamics of ants can be described by the pheromone energy function

$$U(\sigma) = -\ln\left(1 + \frac{\sigma}{1 + \delta\sigma}\right), \quad (26)$$

where σ is the pheromone density, and δ is a dimensionless behavioral parameter.[24] The temperature parameter $T = 1/\beta$ describes the internal randomness of the response of the ants to the pheromonal field. This function is based approximately on a model for Osmotropotaxi (scent gradient following)[3, 4, 5], and on experimental observations of actual ants.[8] For the case where the density of ants is low, and hence the pheromone density is low ($\sigma \ll 1/\delta$), we can make use of the approximate energy function $U_0(\sigma) = -\ln(1 + \sigma)$. The constant $1/\delta$ will be known as the capacity. When σ approaches $1/\delta$ the ants respond less accurately to pheromone gradients.

An illustration of this effect is shown in Figure 3. A given current of organisms I flows into a junction from the left. On the lower branch the pheromone density is fixed at σ_0 , and on the upper branch σ is allowed to vary. $\mathcal{T}(\sigma)$, the proportion of the current which flows into the upper branch, is given by the sigmoidal function

$$\mathcal{T}(\sigma) = [1 + \exp(\beta U(\sigma)/U(\sigma_0))]^{-1}. \quad (27)$$

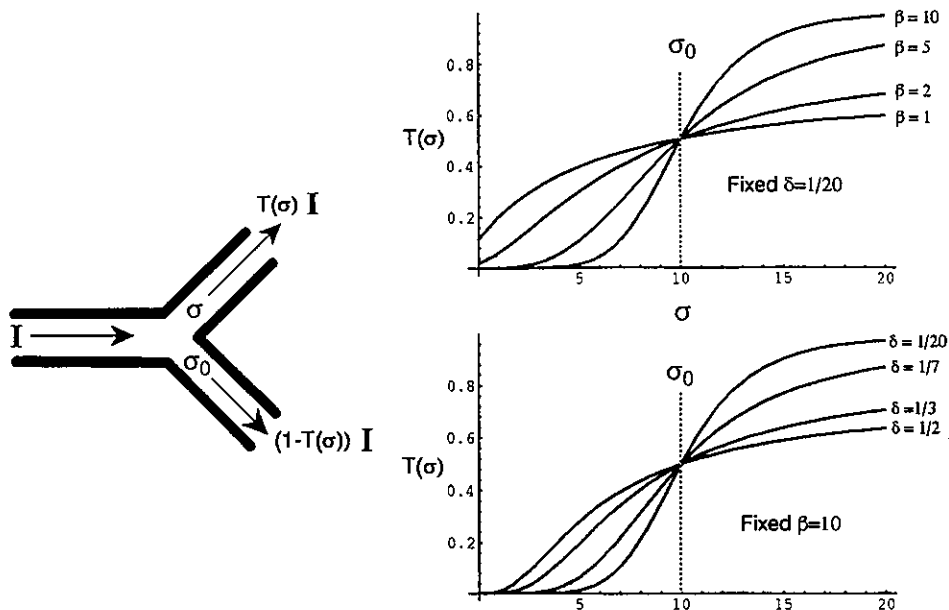


Figure 3: Transition functions for varying β and δ .

The plots on the right of Figure 3 shows $T(\sigma)$ for varying values of β and δ . The upper plot, where δ is fixed, shows the influence of increasing the temperature (lowering β). As the temperature increases the threshold response becomes less and less pronounced. In the opposite limit $\beta \rightarrow \infty$, $T(\sigma)$ would be a step function $\Theta(\sigma - \sigma_0)$. In this limit all of the ants would choose the branch with the greatest pheromone density. In the lower plot the noise level is fixed, and the capacity $1/\delta$ is varied. It is interesting to note that the effects of decreasing the capacity with fixed temperature are similar to the effects of increasing the temperature with fixed capacity. When the density of the ants increases, the pheromone density increases up to and beyond the capacity, the qualitative effects on the behavior of the ants is the same as if the temperature was increased. This gives the swarm roughly the ability to modulate its temperature by modulating its numbers.

This can be made more clear by defining an effective temperature factor $\theta(\sigma)$ through the relation $f(\sigma) = \exp(-\beta U_0(\sigma)/\theta(\sigma))$. $\theta(\sigma)$ roughly measures the effective change in temperature as a function of the pheromonal field when compared to the case where $\delta = 0$, which correspond to the energy

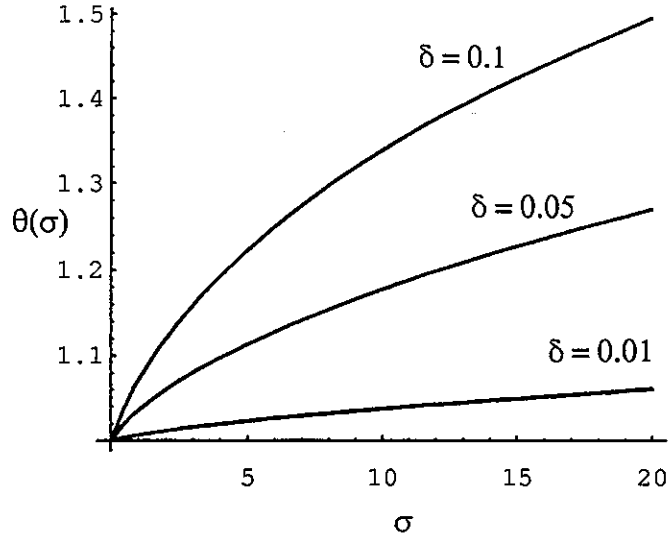


Figure 4: Effective temperature factor

function U_0 . The effective temperature is then given by $\theta(\sigma)T$ where

$$\theta(\sigma) = \frac{\ln\left(1 + \frac{\sigma}{1+\delta\sigma}\right)}{\ln(1+\sigma)}. \quad (28)$$

Fig. 4 illustrates the increase in the effective temperature with increasing σ for three different values of δ . Since increasing the temperature tends to decrease stability, we might expect any organized behavior to breakdown when the number of participants grows too large. It is this ability or the swarm to self-modify its temperature which allows it, in a sense, to traverse its various phase transition boundaries. Such boundaries are of crucial importance in self-organization and emergent phenomena, and it has been proposed that the ability to self-organize at or near these boundaries is at the heart of adaptive, emergent biological systems.[22]

4.2 The phase diagram for the swarm

The condition for stability of the homogeneous phase is independent of v as could be expected, since in the homogeneous phase v does not really exist.

The critical points of the homogeneous phase are given by

$$\gamma_{\star}^{\pm} = \frac{1}{2\delta(1+\delta)} \left\{ \beta - 2\delta - 1 \pm \sqrt{\beta^2 - 2\beta - 4\beta\delta + 1} \right\}. \quad (29)$$

where γ_{\star}^{-} is the value of γ where the symmetric phase becomes unstable as γ is increased, and γ_{\star}^{+} is the value of γ where the symmetric phase again becomes stable. These critical points themselves are the result of a bifurcation controlled by δ . This bifurcation only occurs for delta below the critical point

$$\delta < \delta_{\star} = \frac{(\beta - 1)^2}{4\beta}. \quad (30)$$

When $\delta > \delta_{\star}$ no symmetry breaking is possible, irrespective of γ .

In addition, there is a region of tristability where either the inhomogeneous phase or the homogeneous phase are possible. Which is chosen will depend on the initial conditions, and hysteresis (multiple values of the order parameter for the same values of the state variables) is possible. For a given δ this region extends from $\gamma_{\star}^{+} < \gamma < \gamma_{\star}^0(\nu)$, where γ_{\star}^0 mark the location of a first order transition. For certain case $\gamma_{\star}^0(\nu)$ as a function of δ can be calculated analytically, but we will usually have to resort to Newton's method, or some other numerical scheme. All of this information can be illustrated by plotting the critical points γ_{\star}^{\pm} and $\gamma_{\star}^0(1)$ as a functions of δ . The resulting phase diagram shown in Figure 5 illustrates the regions of symmetric phase, bistability and tristability.

In general the various inhomogeneous states labeled by ν will become unstable at *different values of* γ . In this case we can have a quite complicated sequence of ordering transitions as γ is increased. This ordering can be used to explore some other experiments on ants, but due to the complexity of the subject, this discussion is best taken up elsewhere.

4.3 Branches of stability

We can also investigate the behavior of the various branches of solutions of swarm networks. Figure 6 shows five diagrams, known as bifurcation diagrams, which illustrate the types of behavior as the parameter γ is varied. A bifurcation point is the point where new stable equilibria come into being, that is, a phase transition using our terminology.

Figures 6(A-C) represent the case where $\delta = 0$. Figure 6(A) shows the supercritical bifurcation for $\nu = 1$, the symmetric case. Figure 6(B) shows a subcritical bifurcation for $\nu = 2$. Figure 6(C) represents a special case which

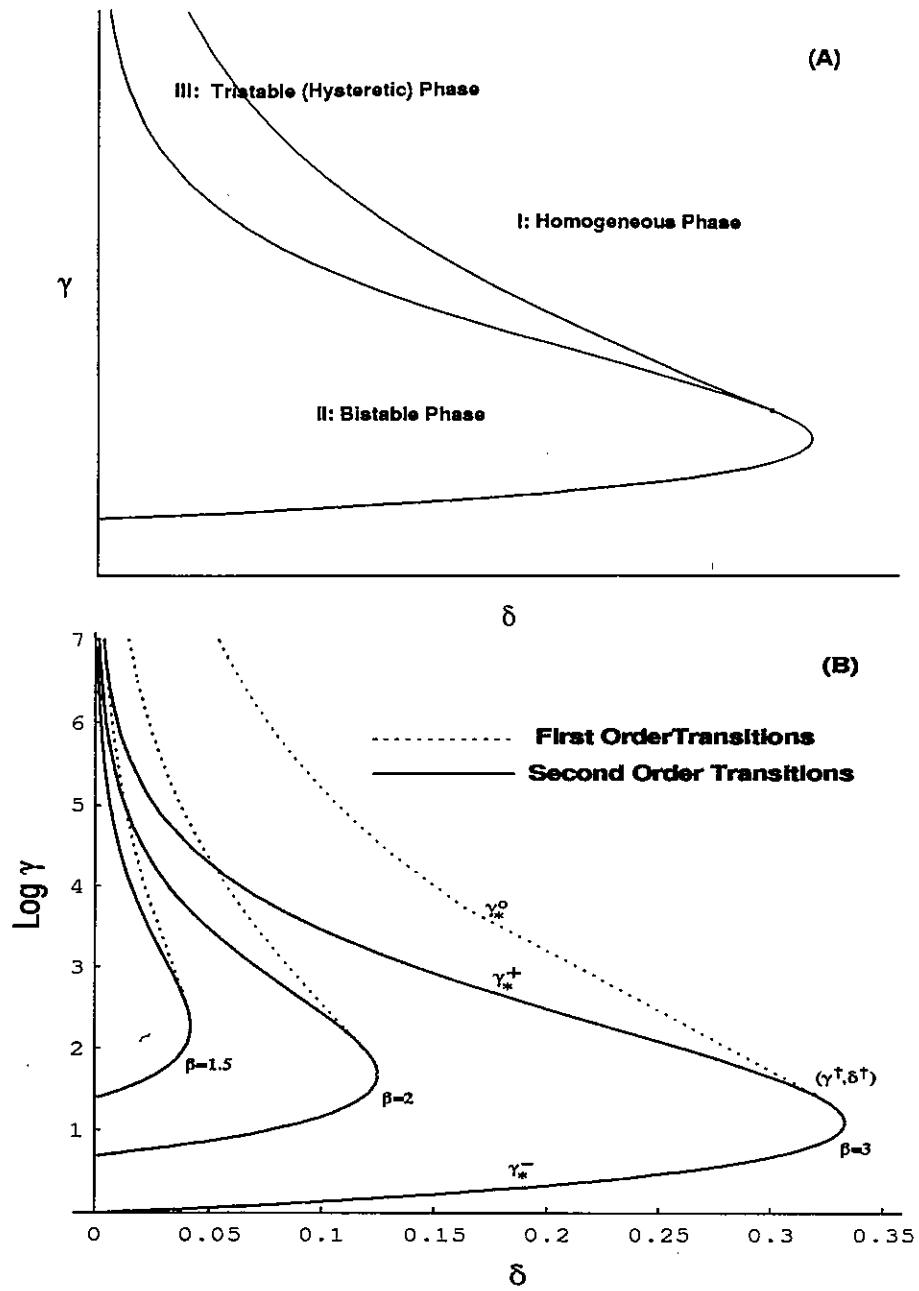


Figure 5: $\delta - \gamma$ phase diagrams.

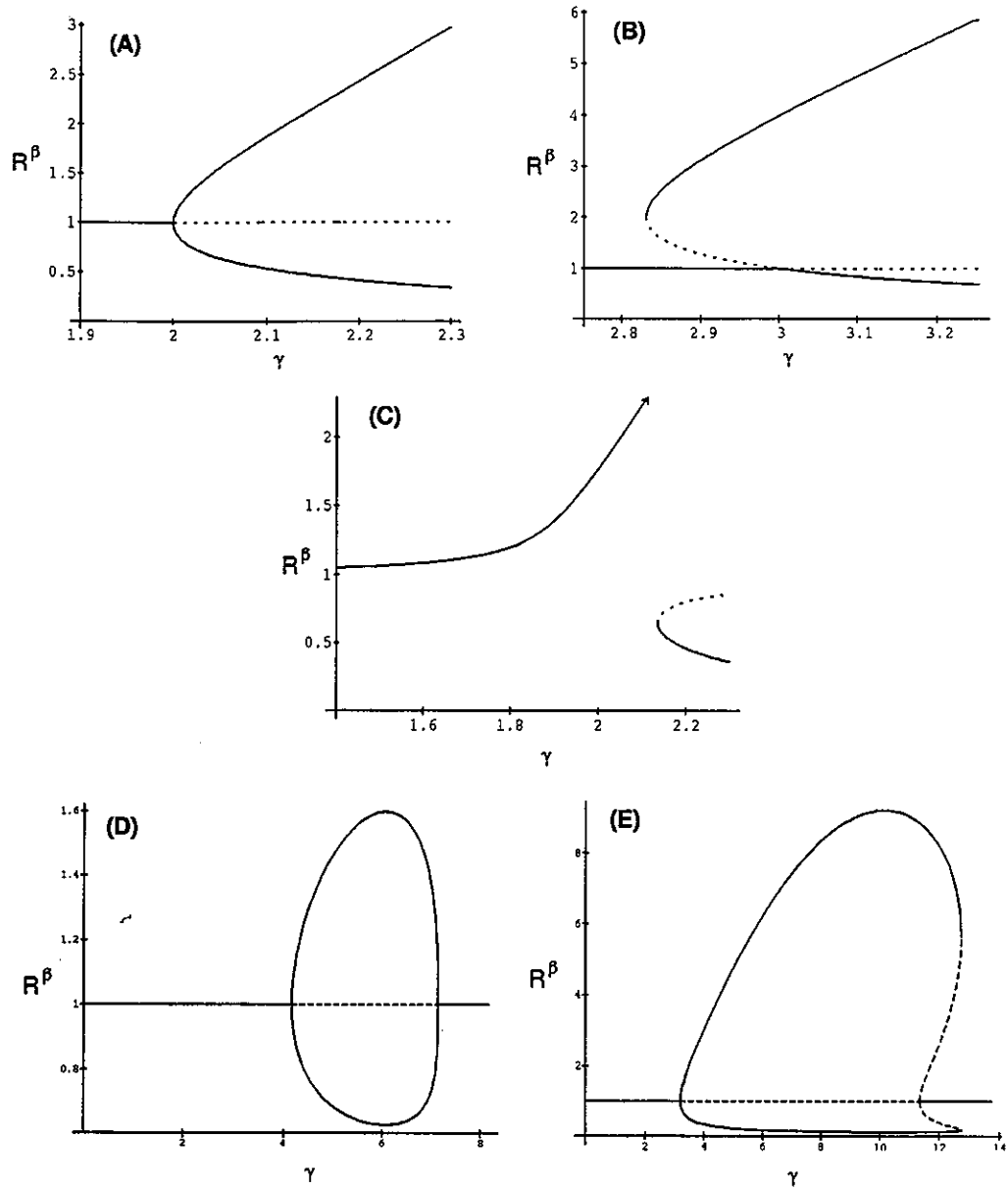


Figure 6: Bifurcation diagrams

will be discussed in the last section of the paper. It shows a transcritical bifurcation for the case where the ants choose between different qualities of food. It is this transcriticality which makes the choice of the better quality food source a more robust aspect of the behavior of swarms than the choice of a shorter segment, as shown by the experimental results mentioned at the end of 1.3.

Figures 6(D-E) show bifurcations for two different values of δ . These should be compared to the equations of state shown in the next section to get a feel for their significance. Both diagrams are for $\nu = 1$. Note that the homogeneous state in Figure 6(E) becomes stable before the broken symmetry states undergo a reverse bifurcation and disappears.

Letting $T = 1/\beta$ be the temperature we can illustrate the phase transition at the critical temperature T_c and the resulting emergence of the order parameter in Figure 7(a). The critical temperature is given, in terms of the other parameters by

$$T_c = \frac{\gamma}{1 + \gamma + 2\delta\gamma + \delta\gamma^2 + \delta^2\gamma^2}. \quad (31)$$

Figure 7(a) is very reminiscent of the magnetization of a substance near its critical temperature.

We can also plot the order parameter as a function of γ , or the equations of state, shown in Figure 7(b), clearly illustrating both the second and first order transitions.

Very close to the critical points, the order parameter scales according to critical exponents which are independent of the particular parameters of the system. We obtain the classical critical behaviors

$$M \sim |T - T_c|^{1/2} \quad (32)$$

$$\chi \sim |\gamma - \gamma_c|^{-1/2} \quad (33)$$

where $\chi = \partial M / \partial \gamma$ is the susceptibility. Figure 8 illustrates this critical behavior.

5 A simple example of swarm behavior

As a basic example of how the attractor structure of the model determines the behavior of a swarm, we will consider the simple cases corresponding to the binary bridge experiments with ants [8, 14] discussed at the beginning of this paper, and compare these cases to the corresponding laboratory experiments.

5.1 Binary node networks

Two basic types of two-segment architecture are illustrated in Figure 9. In case (A) $\mu_1 \neq \mu_2$ and $\eta_1 = \eta_2$. In case (B) we let $\mu_1 = \mu_2$, and $\eta_1 \neq \eta_2$, where we assume, in agreement with experiment, that the ants returning from each food source lay scent at different rates depending on the quality of the food source.[15, 17]

There are four interesting situations which correspond roughly to the cases (1-6) discussed at the beginning of this paper.

- a. Asymmetric double bridge—A with $\mu_1 \neq \mu_2$ which models cases 1 and 2.
- b. Symmetric double bridge—A with $\mu_1 = \mu_2$, which models case 3.
- c. Asymmetric food sources—B with $\eta_1 \neq \eta_2$, which models cases 4 and 5.
- d. Symmetrical food sources—B with $\eta_1 = \eta_2$ which models case 6.

These cases have all been previously explored experimentally, both with actual ants, and with some simple computer simulations.

For this analysis we set $\delta = 0$ and $\beta = 2$ in agreement with the experimentally observed values of these parameters. The fixed points in terms of the ratio of densities on the segments, $S_1/S_2 = R^2$, in various situations are plotted as a function of γ in Figures 6(A-C). The stable fixed points are shown by the solid lines, and the unstable ones by dashed lines. These plots, or bifurcation diagrams, show the values of S_1/S_2 corresponding to the fixed points in scent space. If there is only one stable branch for a given value of γ , then the ant densities will always evolve to a given configuration on the network. If however, there are two stable branches, the system will be forced to choose one or the other. In the absence of noise the system which starts out on one side of the dashed line or the other will always evolve towards the stable state on that side of the line. The natural fluctuations of the density will introduce an element of randomness in the choice of stable state, particularly in those systems which start out on or near the dashed line.

5.1.1 The binary bridge

Symmetry breaking for the binary bridge occurs when $\gamma = 2$. Near a critical point where the $R = 1$ branch becomes unstable one would expect to see a

great increase in the fluctuations of the densities. Figure 10(c) shows an example of these critical point oscillations.

Figures 6(A-B) show the bifurcation diagrams for the symmetric and asymmetric cases respectively. Figure 6(A) illustrates situation **b** described above. The $R = 1$ becomes unstable when $\gamma > 2$, and even though the system is completely symmetric the ants will spontaneously break this symmetry and choose one of the bridges.

Figure 6(B) illustrates what will happen in situation **a**. There are three regions of interest. When $\gamma < 2\sqrt{v}$ only the branch $R = 1$ is possible. When $2\sqrt{v} < \gamma < 1 + v$, $R = 1$ and R_+ , which represents the choice of the shorter branch, are stable. When $\gamma > 1 + v$, R_{\pm} are stable and $R = 1$ is unstable. This says that while the choice of the shorter segment becomes possible before the longer segment, in the absence of noise, the ants should still choose the segment which gets the advantage first.

A full analysis of the effect of noise on the system would be necessary to determine whether the ants would choose the shortest segment with greater probability in this model. It can be shown that noise plays a constructive role in this case.[25] At the point $\gamma(v + 1) = 3$ the $R = 1$ becomes unstable. At this point only R_+ is stable. If the number of ants builds up sufficiently slowly the fluctuations in the system will cause a choice of the shorter segment before, and especially upon, passage through the point $\gamma(1 + v) = 3$.

The choice of the shorter segment can still be understood as a kind of time delayed auto-feedback, as analyzed by Deneubourg et. al.[8] The reasoning proceeds as follows. Ants which take the shorter route reach the food sooner. When they turn around and head for the nest they follow the scent, and they find the scent slightly greater on the route they arrived on, and are thus more likely to choose this route and lay even more scent. Soon the equilibrium densities, which depend on the scents, will be reached, and the auto-feedback stops. However in the short time this has taken the shorter branch has built up a slightly greater scent. This system then effectively starts off above the $R = 1$ line in Figure 6(A). In this case, the ants will usually tend to attract to the shorter branch. From this argument we see how some simple non-equilibrium network properties can be taken into account within the framework of our equilibrium approximations.

This choice of the shorter segment depends on the short-term evolution of the system. However, our calculations show that in the long term, once an equilibrium has been established on a given route, the appearance of a shorter route will not cause the ants to shift to this new route, in agreement with experiment.

5.2 Two food sources

6(C) illustrates the transcritical behavior for the asymmetrical two food source situation c. For a certain range of γ there is only one fixed point, and the ants will always choose the better food source. In this case they will be able to switch to a better food source should one become available. Depending on the scent laying ratio η_1/η_2 there can be quite a large range of γ for which this type of switching is possible. If γ is large enough the choice of the poorer food source also becomes a stable possibility. In this case the ants will be unable to switch to a better food source if offered. It is important to note that if the number of ants on the network slowly builds up from zero, which is what usually happens experimentally, the system will move along the upper branch in figure 6 and will always choose the better food source, no matter what final value γ takes. The case of equal quality food sources is equivalent to the symmetric double bridge case. Provided γ is great enough the ants will choose one of the two sources at random.

5.3 Monte-Carlo simulations

Monte-Carlo simulations of these process were performed to test all of the above predictions, and in order to view the effects of fluctuations on the evolution of the ant densities. In the computer simulations a certain number of ants are allowed to move on the networks. During a time step each ant either moves forward one step on the segment it is on, or if it is at the end of a segment (the segments are all of integer length), chooses a new segment and moves onto it. The scent densities at the ends of each segment are then updated and used to compute the new transition probabilities. The initial configuration of ants and the scent densities on segments are set at the beginning of the program. This representation of the swarm is a probabilistic cellular automata.

In the simulations represented by Figure 10(a) and Figure 10(b), the ants were placed randomly on the network, and the initial pheromone density set to zero. In the situation shown in Figure 10(b) all of the ants started out on segment 2. In these simulations γ was varied by varying the total number of ants on the network. The numbers varied between approximately 400 and 550 ants for the three simulations shown.

Figure 10(a) is a simulation of the binary double bridge, and shows the evolution of the density ratio S_1/S_2 on the network. In this case $\gamma = 2.5$. This is above the critical point $\gamma_c = 2$, so the ants must eventually break

symmetry and attract onto one of the segments. In this case the ants attract onto segment 1. S_1/S_2 hovers around the unstable fixed point at $S_1/S_2 = 1$ until a large enough fluctuation comes along and sends S_1/S_2 towards one of the attractive fixed points. Our previous calculations predict that S_1/S_2 should reach a mean value of 4.00 for this value of γ , and it can be seen that S_1/S_2 reaches a plateau at about this value.

Figure 10(b) illustrates the ability of ants to switch to a better food source. Initially, all the ants are on segment 2 which leads to the poorer food source. The scent laying rate on segment 1 is set about five percent greater than on segment 2, γ is chosen to be just below the point where the bifurcation occurs (see FIG. 7). In this case the ants initially find themselves in an unstable configuration, and it is only a matter of time before they shift to segment 1 where the better food source is located. In the simulation illustrated in Figure 10(b), S_1/S_2 becomes greater than 1 around $t=500$, and reaches a new plateau around $t=900$.

Figure 10(c) again illustrates the symmetric double bridge, this time at the critical point, $\gamma = 2$. The simulation was run for a long time in order to illustrate the long range critical point fluctuations. In most of the simulations S_1/S_2 reaches its fixed point mean value in 1000 to 2000 time steps. Note that when the system is at a critical point it can never make a decision. S_1/S_2 shows fluctuations on an extremely long time scale. These long range correlations are typical of systems at a critical point.

The approach used here is perhaps a little unusual for the study of artificial life. With the availability of computers the usual method of attack is straight simulation. Here, simulations have been performed more or less as illustrations of the theoretical understanding which has been gained. It is hoped that a theoretical attack will not only contribute to understanding in itself, but prove a road map for later more extensive simulations.

6 Summary

In this paper I have tried to show how that it is possible to understand the properties of swarms using some tools from statistical physics. The main goal, that of complete understanding of swarms, was of course left unfinished. However, the more modest goal of setting up a structure which will serve to support such a study, I think has been achieved. We have taken a tour through the mathematical devices which can be used for this purpose, and have introduced a number of new ways to look at the problem in terms

of the vocabulary and physics of phase transitions. The analysis of one type of model gave an indication of the types of things which are possible, as well as ideas for new experiments involving ants. Finally, at the very end, we got a brief glimpse at the complexities which might arise when we will eventually seek to go beyond the deterministic approximation, and include a full understand of the complex effects of fluctuatuions.[25]

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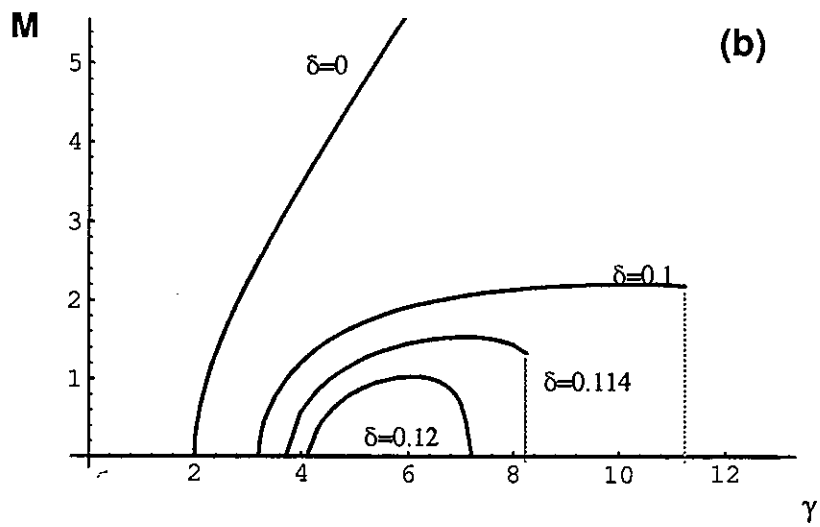
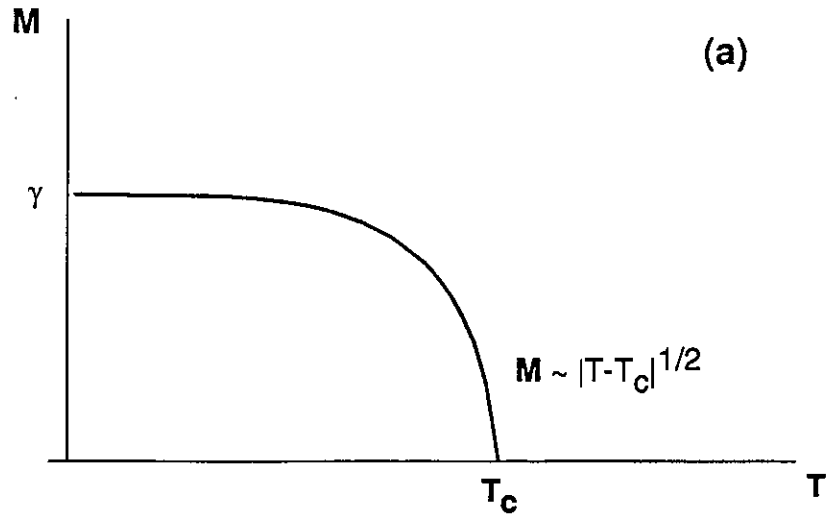


Figure 7: Plots of (a) the order parameter as a function of temperature, and (b) the equations of state for varying values of δ .

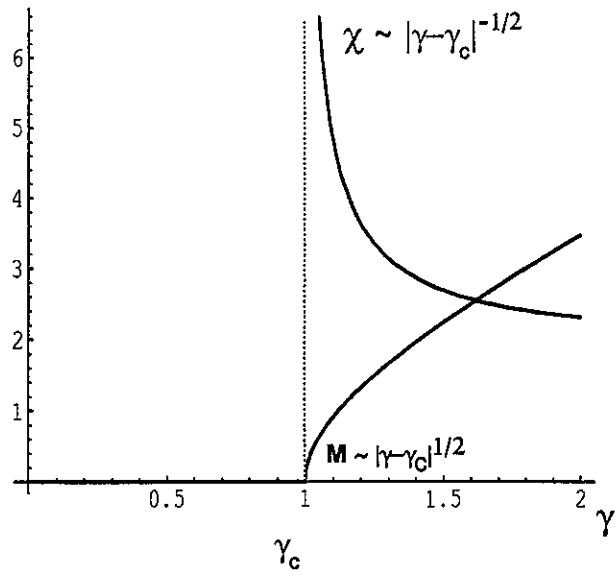


Figure 8: Suseptibility near second order phase transition.

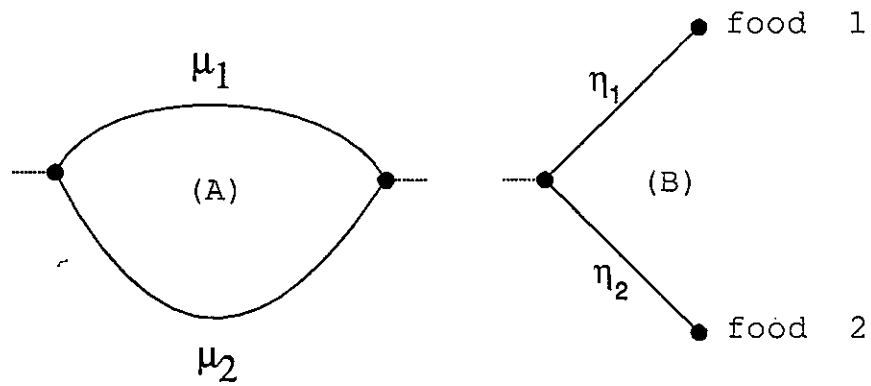


Figure 9: Binary node architecture

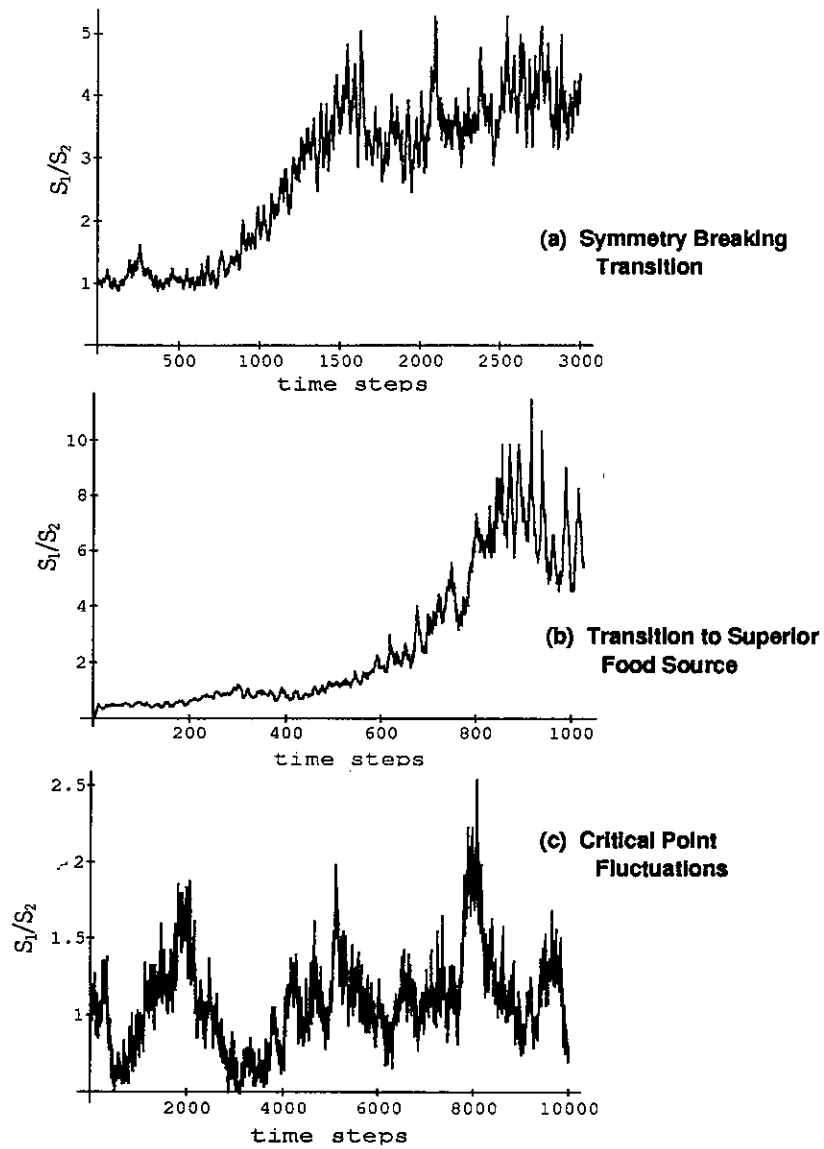


Figure 10: Monte Carlo simulations using a probabilistic cellular automata

A nonequilibrium statistical field theory of swarms and other spatially extended complex systems

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Abstract

A class of models with applications to swarm behavior as well as many other types of spatially extended complex biological and physical systems is studied. Internal fluctuations can play an active role in the organization of the phase structure of such systems. Consequently, it is not possible to fully understand the behavior of these systems without explicitly incorporating the fluctuations. In particular, for the class of models studied here the effect of internal fluctuations due to finite size is a renormalized *decrease* in the temperature near the point of spontaneous symmetry breaking. We briefly outline how these models can be applied to the behavior of an ant swarm.

In this paper I introduce a class of models which is in line with the basic processes acting in a variety of systems in nature, particularly biological ones. Some systems which fall into this class are insect swarms, swimming bacteria and algae,[6] physical trail formation, the evolution of river networks,[7] diffusive transport in polymeric materials,[1] population distribution models, various types of fractal growth phenomena,[13] and developmental morphogenesis.[11]

Here we study what will be called *stigmergic processes* as a generalization of the concept of stigmergy introduced by Grassé[3] in the context of collective nest building in social insects. The hypothesis of stigmergy, as described by Wilson[14], is that *it is the work already accomplished, rather than direct communication among nest mates, that induces the insects to perform additional labor*. The concept of stigmergy has also been invoked more recently in regards to swarm behavior.[12]

The more generalized idea of a stigmergic process is realized here in systems composed of three basic ingredients. The first ingredient is a *particle dynamics* which obeys a Markov process on some finite state space \mathcal{X} . The particle density $\rho(\mathbf{x}, \tau)$ obeys the Master equation

$$\frac{\partial \rho(\mathbf{x}, \tau)}{\partial \tau} = \int_{\mathcal{X}} \{W_{\tau}(\mathbf{x}|\mathbf{y})\rho(\mathbf{y}, \tau) - W_{\tau}(\mathbf{y}|\mathbf{x})\rho(\mathbf{x}, \tau)\} d^D \mathbf{y}, \quad (1)$$

where $W_{\tau}(\mathbf{x}|\mathbf{y})$ is the probability density to go from state \mathbf{y} to \mathbf{x} at time τ . The second element is a *morphogenetic field* $\sigma(\mathbf{x}, \tau)$, representing the environment which the particles both respond to, and act on. We will study one of the simplest situations, a fixed one-component pheromonal field which evolves according to

$$\frac{\partial \sigma(\mathbf{x}, \tau)}{\partial \tau} = -\kappa \sigma + \eta \rho, \quad (2)$$

where κ measures the rate of evaporation, breakdown or removal of the substance, and η the rate of emission of the pheromone by the organisms. Lastly, some form of *coupling* is made between the particles and the field. This coupling takes the form of a behavioral function which describes how the particles move in response to the morphogenetic field, and in turn, how the particles act back on this field.

As we shall see, small changes in the microscopic behavior of the particles can result in large changes in the global behavior of the swarm, or particle field. This variability has significant implications not only for the behavioral response of the swarm to external stimuli, but also in the evolution of cooperative behavior. Wilson has remarked that an understanding of how this occurs would *constitute a technical breakthrough of exciting proportions, for it will then be possible, by artificially changing the probability matrices, to estimate the true amount of behavioral evolution required to go from [the behavior of] one species to ... that of another.*[14] He has further remarked that such large behavioral changes resulting from small changes in the individual dynamics would provide evidence that social behavior evolves at least as rapidly as morphology in social insects. This could provide an explanation why *behavioral diversity far outstrips morphological diversity at the level of species and higher taxonomic categories* in social insects.

In the region of a nonequilibrium phase transition the morphogenetic field, and hence the transition matrix, changes very slowly on scales typical of the particle field relaxation time since in this region the unstable modes will exhibit critical slowing down and will relax on a time scale much longer than the time scale of the stable modes. The particle modes are said to be

slaved to the morphogenetic field, and can be adiabatically eliminated from the picture.[4] We obtain the stochastic order parameter equation

$$\frac{\partial \sigma(\mathbf{x}, \tau)}{\partial \tau} = \kappa \sigma + \eta \rho_s[\sigma] + \eta g[\sigma] \xi(\mathbf{x}, \tau), \quad (3)$$

where $\rho_s[\sigma]$ is the quasi-stationary particle density, $g[\sigma]$ is a function describing the fluctuations of the quasi-stationary particle density about its mean value, and $E\{\xi(\mathbf{x}, \tau)\} = 0$, $E\{\xi(\mathbf{x}, \tau)\xi(\mathbf{x}', \tau')\} = \delta(\mathbf{x} - \mathbf{x}')\delta(\tau - \tau')$. Since ρ_s will depend on both the global state of the morphogenetic fields, and on the global boundary conditions, this is a globally coupled set of equations for the evolution of the morphogenetic fields. *Slaving of the particle field therefore allows an explicitly coupled global dynamics to emerge from the strictly local interactions of the model*, providing a key to how a globally integrated response may emerge from a system of locally acting agents.

The the fluctuations in the system are state dependent. In addition to amplifying an instability which exists in the absence of noise, this type of fluctuation can also produce transitions and ordered behavior in its own right. One of the consequence of this fact is that slaved particle field will constructively determine the self-organization properties of the systems *through its fluctuating properties*, as well as through quasi-stationary values. This is a fact which should be constantly be born in mind when studying such models.

For the purposes of this paper we will consider the case where the transition matrix takes the form $W(\mathbf{x}|\mathbf{y}) \propto f(\sigma(\mathbf{x}))p(|\mathbf{x} - \mathbf{y}|)$, where f is some weighting function describing the effect of the field σ on the motion of the particles, and $p(|\mathbf{x} - \mathbf{y}|)$ is a probability distribution of jumps of length $r = |\mathbf{x} - \mathbf{y}|$. Transition matrices of this type obey detailed balance, $W(\mathbf{x}|\mathbf{y})f(\sigma(\mathbf{y})) = W(\mathbf{y}|\mathbf{x})f(\sigma(\mathbf{x}))$. In this case we can define a partition function

$$Z = \left\{ \frac{1}{V} \int d^D \mathbf{x} f(\sigma(\mathbf{x})) \right\}^N, \quad (4)$$

where V is the volume of the state space \mathcal{X} , and N is the total number of particles. A one-to-one analogy with a thermodynamic system with energy $U(\sigma(\mathbf{x}))$ and temperature $T = \beta^{-1}$ can be made if we set $f(\sigma(\mathbf{x})) = \exp(-\beta U(\sigma(\mathbf{x})))$, where any parameter T can be regarded as a temperature parameter if $f(\sigma(\mathbf{x}); \alpha T) = f^{-\alpha}(\sigma(\mathbf{x}); T)$. Statistical quantities of interest can be calculated from the partition function according to the usual prescriptions. In a closed system the mean particle density and dispersion in

the energy state ϵ are given by

$$E\{\rho_\epsilon\} = \frac{N}{VZ} \exp(-\beta\epsilon), \quad E\{(\Delta\rho_\epsilon)^2\} = \frac{E\{\rho_\epsilon\}}{\mu_\epsilon} \left(1 - \frac{\mu_\epsilon}{N} E\{\rho_\epsilon\}\right), \quad (5)$$

where μ_ϵ is the volume of the system in energy state ϵ . The slaved particle field in energy state ϵ can then be represented, to lowest order in the fluctuations, by $\rho_\epsilon[\sigma] = E\{\rho_\epsilon[\sigma]\} + \sqrt{E\{(\Delta\rho_\epsilon)^2[\sigma]\}} \xi(\mathbf{x}, t)$.

We introduce the dimensionless parameter $\bar{\rho} = N/V$, the mean density of particles, and $\nu = \mu^-/\mu^+$, the ratio of the volume of the field $\sigma(\mathbf{x})$ in the σ^- state to the volume in the σ^+ state. We also define the function $R(\sigma^+, \sigma^-) = f(\sigma^+)/f(\sigma^-)$. In the mean field approximation a Langevin equation

$$\frac{dm}{dt} = -m + F(m) + \frac{1}{\sqrt{N}} Q(m) \xi(t) \quad (6)$$

for the order parameter m can be derived,[10] where

$$F = \bar{\rho}(1+\nu) \frac{R^\beta - 1}{R^\beta + \nu}, \quad Q^2 = \frac{\bar{\rho}^2(1+\nu)^4}{\nu} \frac{R^\beta}{(R^\beta + \nu)^2}, \quad (7)$$

and where the F and Q are determined as functions of m by

$$R(m) = R\left(\bar{\rho} + \frac{\nu m}{1+\nu}, \bar{\rho} - \frac{m}{1+\nu}\right). \quad (8)$$

The order parameter m is analogous to a gas-liquid order parameter, and represents the difference in the values of the field in the σ^+ and σ^- states after spontaneous symmetry breaking. The behavior of this system is described by the potential function

$$\Phi(m) = \int^m \frac{m - F(m)}{Q^2(m)} dx + \frac{1}{N} \ln Q, \quad (9)$$

where the phases m_i of the system are determined by the conditions $\Phi'(m_i) = 0$, $\Phi''(m_i) > 0$. [5]

In the continuum limit ($N \rightarrow \infty$) it can be shown that the critical value of the mean density $\bar{\rho}_c$ at which spontaneous symmetry breaking occurs is given by the condition $-\bar{\rho}_c U'(\bar{\rho}_c) = T$. Generally $\bar{\rho}_c$ will increase with increasing temperature. The relative stability of two phases m_1 and m_2 is determined by the relative potentials $\Phi(m_1)$ and $\Phi(m_2)$ for each phase. Even in the continuum limit the details of the fluctuations cannot be neglected due

to the presence of the factor $Q^2(m)$ under the integral in 9, and the relative stability of the phases will depend on the precise details of the internal fluctuations. Similar observations have been made elsewhere by Landauer and others.[8]

When N is finite, the situation is still more complicated. It is clear that the possible values of the order parameter and the phase structure do not remain unchanged under the influence of internal fluctuations. The criterion for spontaneous symmetry breaking in this case is $-\bar{\rho}_c U'(\bar{\rho}_c) = \hat{T}$, where \hat{T} is the renormalized temperature $\hat{T} = \gamma(N)T$ where $\gamma(N) = \sqrt{N + (N/2)^2} - N/2$. This is precisely the continuum condition except that the finite size fluctuations have the effect of renormalizing the temperature by the factor $\gamma(N) < 1$. The effect of increasing the internal fluctuations through decreasing the total number of particles has the effect of *decreasing the temperature*. We thus arrive at the seeming paradox that increased internal fluctuations may produce increased order.

I will now briefly outline how the previous analysis can be applied to the example of an ant swarm. More details can be found elsewhere.[9] In this case the individual ants are the particles, and the morphogenetic field is a pheromonal substance which the ants sense with their antennae, and emit from their bodies as they move. The basic measurement the ants make is the quantity of pheromone receive by each antennae. They can therefore respond to difference in the pheromone between the antennae, and move accordingly. A very general model of such motion assume that the particle experience a *force* which is proportional to the scent gradient at that point multiplied by some nonlinear response function $\chi(\sigma)$ of the scent at that point. The nonlinear response function models the nonlinearities underling the basic physiology of the sensing apparatus, for instance, any nonlinear neural/receptor response to the pheromone, including such effects as saturation of the receptor sites on the antennae by the pheromonal substance. In addition there is an element of randomness due to fluctuations in the external environment as well as internal fluctuations. These are incorporated into an effective random force with a strength in proportion to \sqrt{T} where T is the temperature factor. The motion of a particle can be described by a Langevin equation of the form

$$\frac{d\mathbf{x}(t)}{dt} = \chi(\sigma(\mathbf{x}))\nabla\sigma + \sqrt{2T} \xi(t), \quad (10)$$

where $E\{\xi(t)\} = 0$, and $E\{\xi(t)\xi(t')\} = \delta(t-t')$. This can be written in the

form

$$\frac{dx(t)}{dt} = -\nabla U(\mathbf{x}) + \sqrt{2T} \xi(t), \quad (11)$$

where $\chi(\sigma(\mathbf{x})) = -U'(\sigma(\mathbf{x}))$. Easy to show that the behavioral function of such a system is given by $f(\sigma) = \exp(-\beta U(\sigma))$.

The microscopic dynamics of the ants which we will study in the rest of the paper is determined by the response function

$$\chi(\sigma) = \alpha + \frac{c\rho}{c + \rho}, \quad (12)$$

where ρ is the pheromone density, and α and c are constants with the units of pheromone density. This function is inspired by the observed behavior of actual ants[2]. The constant α is roughly the threshold where the response of the ants to the pheromone is small unless $\rho > \alpha$. The constant c will be known as the capacity. When ρ approaches c the ants respond less accurately to pheromone gradients. This is because when the pheromone density is very large the antennae receptors become saturated and the ant can not sense the pheromone gradient as accurately.

For simplicity we will introduce the dimensionless variable $\sigma = \rho/\alpha$ and the dimensionless parameter $\delta = \alpha/c$, where $1/\delta$ is the dimensionless capacity. The energy function takes the form

$$U(\sigma) = -\ln\left(1 + \frac{\sigma}{1 + \delta\sigma}\right), \quad (13)$$

where we drop off any additive constant term, which have no effect on the behavior of the ants. For the case where the density of ants is low, and hence the pheromone density is low ($\rho \ll g$), we can make use of the approximate energy function $U_0(\sigma) = -\ln(\alpha + \sigma)$.

An illustration of this effect is shown in Figure 1. A given current of organisms I flows into a junction from the left. On the lower branch the pheromone density is fixed at σ_0 , and on the upper branch σ is allowed to vary. $\mathcal{T}(\sigma)$, the proportion of the current which flows into the upper branch, is given by the sigmoidal function

$$\mathcal{T}(\sigma) = [1 + \exp(\beta U(\sigma)/U(\sigma_0))]^{-1}. \quad (14)$$

The plots on the right of Figure 1 shows $\mathcal{T}(\sigma)$ for varying values of β and δ . The upper plot, where δ is fixed, shows the influence of increasing the temperature (lowering β). As the temperature increases the threshold response becomes less and less pronounced. In the opposite limit $\beta \rightarrow \infty$,

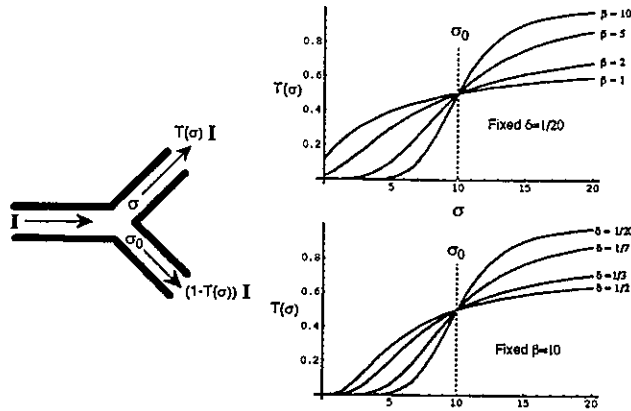


Figure 1: Transition functions for varying β and δ .

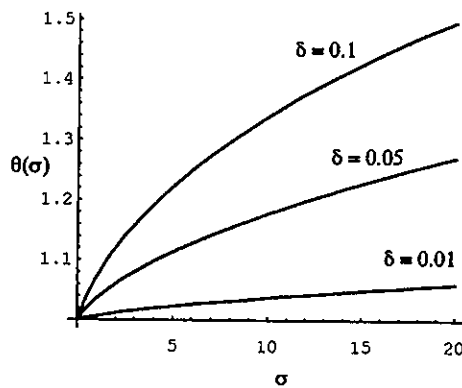


Figure 2: Effective temperature factor

$\mathcal{T}(\sigma)$ would be a step function $\Theta(\sigma - \sigma_0)$. In this limit all of the ants would choose the branch with the greatest pheromone density. In the lower plot the noise level is fixed, and the capacity $c = \alpha/\delta$ is varied. It is interesting to note that the effects of decreasing the capacity with fixed temperature are similar to the effects of increasing the temperature with fixed capacity. When the density of the ants increases, the pheromone density increases up to and beyond the capacity, the qualitative effects on the behavior of the ants is the same as if the temperature was increased. This gives the swarm roughly the ability to modulate its temperature by modulating its numbers.

This can be made more clear by defining an effective temperature factor $\theta(\sigma)$ through the relation $f(\sigma) = \exp(-\beta U_0(\sigma)/\theta(\sigma))$. $\theta(\sigma)$ roughly mea-

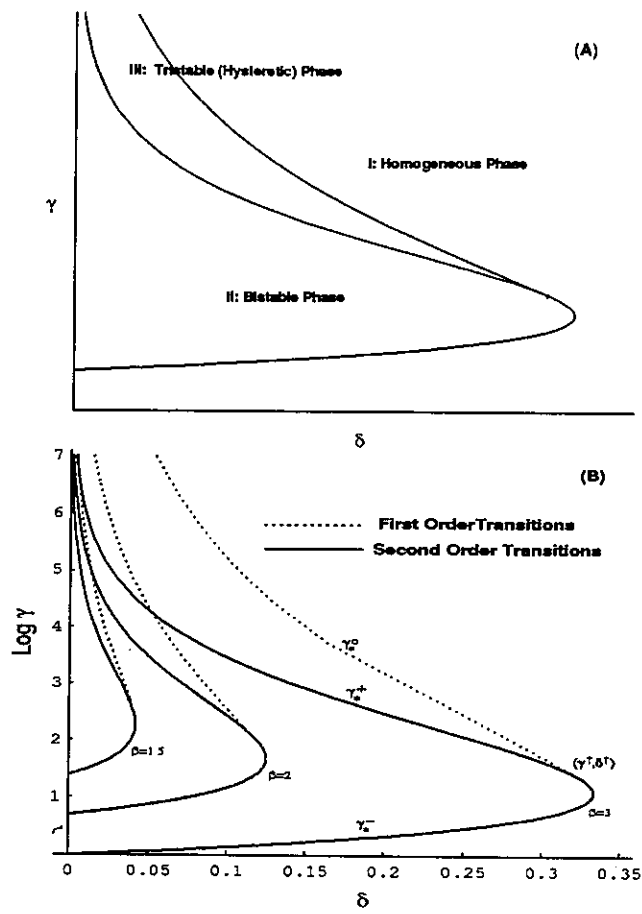


Figure 3: $\bar{\rho} - \delta$ phase diagram for the ant swarm.

asures the effective change in temperature as a function of the pheromonal field when compared to the case where $\delta = 0$, which correspond to the energy function U_0 . The effective temperature is then given by $\theta(\sigma)T$ where

$$\theta(\sigma) = \frac{\ln\left(1 + \frac{\sigma}{1+\delta\sigma}\right)}{\ln(1 + \sigma)}. \quad (15)$$

Fig. 2 illustrates the increase in the effective temperature with increasing σ for three different values of δ . Since increasing the temperature tends to decrease stability, we might expect any organized behavior to breakdown when the number of participants grows too large. It is this ability of the swarm to self-modify its temperature which allows it, in a sense, to traverse its various phase transition boundaries.

Figure 3 is a typical phase plot for the ant swarm illustrating regions of homogeneity, bistability and hysteresis. The plot illustrates the effect of behavioral and swarm parameters on the swarm as a whole. In this case δ is a behavioral parameter which could be expected to change on the evolutionary time scale, and γ , which is proportional to the number of participants, is a swarm parameter which determines the behavioral "phase" of the swarm. More details may be found in previously published papers[9] where the properties of an ant swarm are analyzed in depth, and it is also shown how the collective behavior of real ants[2] can be understood in terms of such models.

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